Tutorials and worked examples for simulation, curve fitting, statistical analysis, and plotting.
https://simfit.org.uk
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The great importance of the t distribution in data analysis lies in the existence of numerous tests based upon it, such as the 1-sample t, unpaired t, and paired t, as well as the use in calculating confidence intervals.

1 Definitions

Consider two independent random variables, Z which has a normal distribution with $\mu = 0$, $\sigma^2 = 1$, and C which has a chi-square distribution with k degrees of freedom. Then the ratio

$$t_k = \frac{Z}{\sqrt{C/k}}$$

is described as a t variable with k degrees of freedom. It should be noted incidentally that t_k^2 is distributed as $F(1, k)$.

A special case arises when analyzing a sample of size n from a normal distribution with population mean μ and population variance σ^2, because the sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

is normally distributed with mean μ and variance σ^2/n, while nS^2/σ^2 using

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

has a χ^2 distribution with $n-1$ degrees of freedom. Hence the statistic

$$t_{n-1} = \frac{\bar{x} - \mu}{S/\sqrt{n-1}}$$

has a t distribution with $n-1$ degrees of freedom. Note that this t variable only has one unknown parameter, the population mean μ.

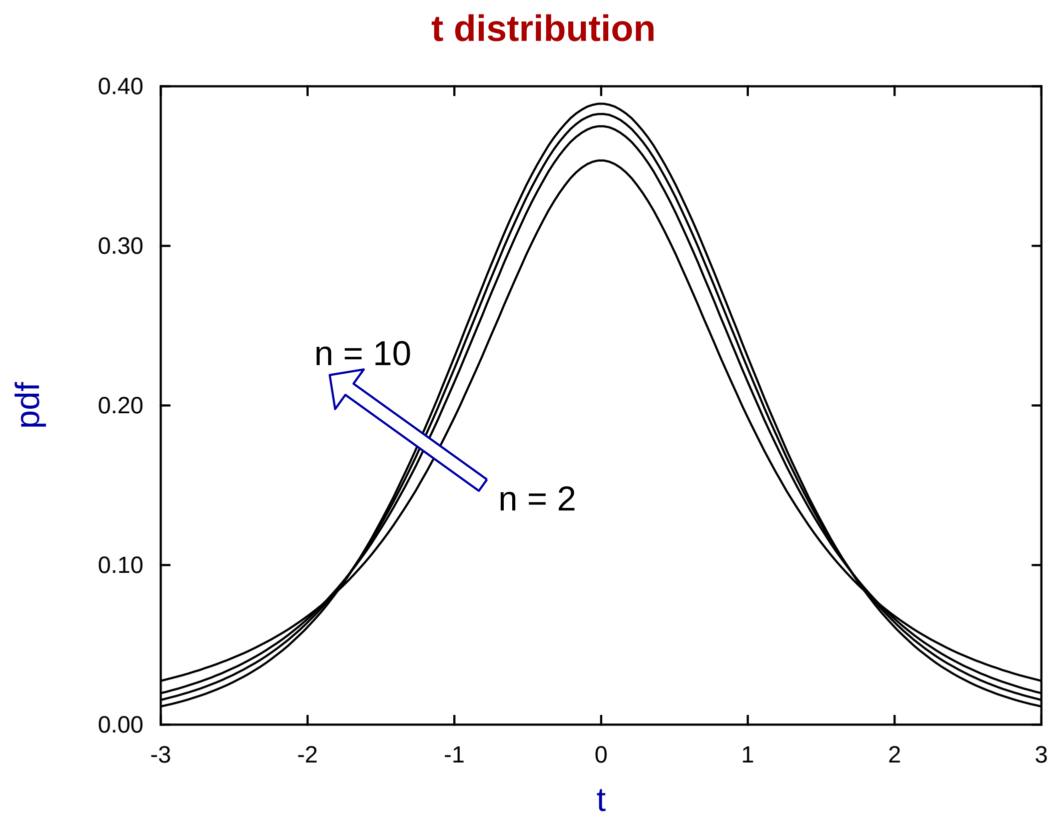
2 Simfit program ttest

Choose `[A/Z]` from the main SimFIT menu and open program **ttest** when the following options will be available.

```
Input: N, number of degrees of freedom
Input: t, calculate pdf(t)
Input: t, calculate cdf(t)
Input: alpha, calculate t inverse
Input: data, 1-sample t test
Input: data, 2-sample unpaired t test
Input: data, 2-sample paired t test
Input: matrix, groups across rows t test
Power and sample size
Non-central t distribution.
```

3 Degrees of freedom

An important use of the t distribution is when calculating confidence limits, for instance with a sample mean, or parameter estimate. The main thing to realize in such circumstances is that, although the mean value for t_n is zero irrespective of n, the variance is heavily dependent on n. This is why the confidence limits shrink as the sample size increases. Actually the t_n distribution is asymptotic to a standardized normal distribution as n increases, as shown by the next graph created from **ttest**.



Note how the area under the tails decreases rapidly as n increases from 2 to 6 but less slowly thereafter. A more detailed inspection of this will be clear from this table copied from the **ttest** results log file for a 95% confidence interval.

```
P(t =< 4.303E+00) = 0.975 *** P(t >= 4.303E+00) = 0.025, N = 2
P(t =< 2.776E+00) = 0.975 *** P(t >= 2.776E+00) = 0.025, N = 4
P(t =< 2.447E+00) = 0.975 *** P(t >= 2.447E+00) = 0.025, N = 6
P(t =< 2.306E+00) = 0.975 *** P(t >= 2.306E+00) = 0.025, N = 8
P(t =< 2.228E+00) = 0.975 *** P(t >= 2.228E+00) = 0.025, N = 10
```

4 Confidence range for the sample mean

Given $\bar{x}$ and S^2 from a sample of size n, then a symmetrical $100(1-\alpha)\%$ confidence range for the population mean μ can be constructed using the upper tail critical value $t_{\alpha/2,n-1}$. We have that

$$P\left(\frac{\bar{x}-\mu}{S/\sqrt{n-1}} \geq t_{\alpha/2,n-1}\right) = \alpha/2$$

and

$$P\left(\frac{\bar{x}-\mu}{S/\sqrt{n-1}} \leq -t_{\alpha/2,n-1}\right) = \alpha/2,$$

so that

$$P\left(\bar{x} - t_{\alpha/2,n-1}S/\sqrt{n-1} \leq \mu \leq \bar{x} + t_{\alpha/2,n-1}S/\sqrt{n-1}\right) = 1-\alpha.$$

Alternatively, note that it often causes confusion because an unbiased estimate of the population variance is not S^2 but the sample variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x}\ )^2,$$

so that an equivalent expression for t_{n-1} would then be

$$t_{n-1} = \frac{\bar{x}-\mu}{s/\sqrt{n}},$$

whereupon

$$P\left(\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \leq \mu \leq \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n}\right) = 1-\alpha.$$

using s^2 instead of S^2.

We see from the above table that the multipliers of the sample standard error required for a 95% confidence interval with sample sizes of $n = 3, 5, 7, 9,$ and 11 would be 4.303, 2.776, 2.447, 2.306, and 2.228. Clearly, using the sample mean plus or minus twice the standard error as an approximate 95% confidence range will always underestimate the actual 95% confidence range unless the sample size exceeds 10, say.